Selected Works of Daniel Cristofaro-Gardiner

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In This Talk

- From One to Two or Infinitely Many Reeb Orbits.
- Symplectic Weyl Law.
- A Global Le Calvez-Yoccoz Property.
- Packing Stability.

The Idea of Symplectic Dynamics

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Moser and Zehnder—The Dynamical Systems Workshops at Oberwolfach
Katok

A Short History of Symplectic Topology

Here are some of the researchers who played a signficant role initially on the symplectic side.

Names who prepared the ground are:

Vladimir Arnold and Alan Weinstein—developing a symplectic framework and asking guiding questions in the seventies.

Important events and milestones which ultimately started up the whole development are due to

Paul Rabinowitz (1977), Charles Conley and Edi Zehnder (1983), Misha Gromov (1985) culminating in the work of Andreas Floer (1985-1990).

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Meanwhile we have many different implementations—addressing a broad array of problems.

Forty Years Later—The Magic

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Action principles for special types of orbits.

Combinations of these.

Birth-death-type phenomena.

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YES WE CAN!

Filtered Chain Complexes—Enriched in Various Ways

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Filtered Chain Complexes—Enriched in Various Ways

$$\ldots \Longrightarrow \boxed{\mathfrak{J}_n} \Longrightarrow \boxed{\mathfrak{J}_{n-1}} \Longrightarrow \ldots \boxed{\mathfrak{J}_j} \ldots \Longrightarrow \boxed{\mathfrak{J}_0} \ldots \Longrightarrow \boxed{\mathfrak{J}_{-k}} \ldots$$

Forty Years Later—The Magic

Given a physical system one often succeeds to view collections of special solutions as some kind of "critical point" of some function.

Then one studies solutions of associated partial differential equations which define geometric objects which can be viewed as "gradient lines" between "critical points".

Then all this information can be packaged as an object in homological algebra. In general, deformations of our system will lead to homotopies between the algebraic objects. A lot of good things are happening.

Another interesting fact is that in Morse theory there is some C^0 stability. This has a lot of consequences. For example the definition of what is a symplectic map requires one derivative. However, this stability implies the existence of a C^0-symplectic geometry.

The work of Daniel Cristofaro-Gardiner

- From One to Two or Infinitely Many Reeb Orbits.
- Symplectic Weyl Law
- A Global Le Calvez-Yoccoz Property.
- Packing Stability.

From One to Two or Infinitely Many Reeb Orbits

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We need to know for the following:

Y closed smooth manifold of dimension $2n+1$.
λ contact form: $\lambda \wedge d\lambda^n =$ volume-form.
X Reeb vector field: $i_X\lambda = 1$ and $i_X d\lambda = 0$.

In the following $\dim(Y) = 3$.

Two or Infinitely Many Reeb Orbits.

Theorem (Cristofaro-Gardiner, Hryniewicz, Hutchings, Liu (2023))

Let Y be a closed connected three-manifold and λ a contact form on Y. Assume that the associated contact structure $\xi = \ker(\lambda)$ has a first Chern class $c_1(\xi) \in H^2(Y, \mathbb{Z})$ which is torsion. Then the Reeb vector field associated to λ either has two or infinitely many simple periodic orbits.

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No degeneracy assumption!

Finsler metrics or Riemannian metrics on S^2 provide Reeb flows on the unit-sphere bundle of S^2.

Katok's example with two geodesics.

Bangert-Franks-Hingston: Infinitely many closed prime geodesics on a Riemannian S^2 with a growth property.

Precisely Two

Theorem (Cristofaro-Gardiner, Hryniewicz, Hutchings, Liu (2021))

If the connected compact three-manifold Y equipped with a contact form has precisely two periodic orbits, then both periodic orbits are non-degenerate. Further the three-manifold Y is diffeomorphic to the three-sphere or a Lens space. The two periodic orbits are the core circles of a Heegaard genus-one splitting and the the Reeb flow has a global disk-like surface of section with its dynamic being a pseudo rotation. In the sphere case the periods and rotation numbers of the simple Reeb orbits behave like an irrational ellipsoid.

Two or infinitely many in the non-degenerate case

Theorem (Cristofaro-Gardiner, Hutchings, Pomerleano (2017))

Let Y be a closed connected three-manifold and λ a non-degenerate contact form on Y. Assume that the associated contact structure $\xi = \ker(\lambda)$ has a first Chern class $c_1(\xi) \in H^2(Y, \mathbb{Z})$ which is torsion. Then the Reeb vector field associated to λ either has two or infinitely many simple periodic orbits.

Two or infinitely many in the non-degenerate case

Theorem (Cristofaro-Gardiner, Hutchings, Pomerleano (2017))

Let Y be a closed connected three-manifold and λ a non-degenerate contact form on Y. Assume that the associated contact structure $\xi = \ker(\lambda)$ has a first Chern class $c_1(\xi) \in H^2(Y, \mathbb{Z})$ which is torsion. Then the Reeb vector field associated to λ either has two or infinitely many simple periodic orbits.

A nontrivial input to the proof is a symplectic Weyl law, which is also the starting point for smooth closing lemmas. Assuming there are only finitely many periodic orbits, the proof uses pseudoholomorphic curve theory to construct a genus zero surface of section and then uses Franks' theorem to derive a contradiction.

Here we have a connection with the work of Hofer-Wysocki-Zehnder (1998, 2003) using holomorphic curve theory for contact manifolds (Hofer 1993) to construct global surfaces of sections or generalizations thereof in the star-shaped case in $\mathbb{R}^4$.

At least two

Theorem (Cristofaro-Gardiner, Hutchings (2012))

Any closed connected contact three-manifold (Y, λ) has at least two embedded Reeb orbits.

The main point is not to assume non-degeneracy. It uses a delicate argument involving the symplectic Weyl formula.
Taubes (2007) already proved for every (Y, λ) there is at least one closed Reeb orbit (Weinstein Conjecture in dimension three (open in higher dimensions)). Hofer (1993) for $Y = S^3$, or if $\pi_2(Y) \neq 0$ or the contact form overtwisted.

Taubes "secret sauce" is the relationship between SWF and ECH.

Weyl Formula

As I was told the Weyl formula came out of the research to obtain the previous result.

Theorem (Cristofaro-Gardiner, Hutchings, Ramos (2012))

Pick any sequence (σ_k) of homology classes in $ECH(Y)$ such that $grd(\sigma_k) \to \infty$. Denote by $c_\lambda(\sigma_k)$ the infimum of all L such that σ_k is in the image of the map $E^L(Y, \lambda, J) \to E(Y)$. Then

$$\lim_{k\to\infty} \frac{(c_\lambda(\sigma_k))^2}{grd(\sigma_k)} = 2\int_Y \lambda \wedge d\lambda.$$

The impact of the Weyl fomula was remarkable. Kei Irie used it to prove a smooth closing lemma for Reeb flows and for Hamiltonian diffeomorphism on surfaces. Dan Cristofaro-Gardiner and and Rohil Prasad recently proved the most general result and Bassam will talk about it.

Global LeCalvez-Yoccoz Property

In its classical form: Le Calvez and Yoccoz proved in 1997 that for any homeomorphism of the 2-sphere, the complement of a finite invariant set is never minimal. Here, a set is minimal if every orbit in the set is dense in it. In other words, you cannot find a proper compact invariant set whose complement is minimal for the dynamics; there will always be points whose orbits are not dense in the complement.

In a recent paper (2023) by Fish-Hofer it was shown that one can use pseudoholomorphic to find closed invariant sets beyond periodic orbits and a conjecture by Michel Herman in dimension four was proved, namely that on a compact regular Hamiltonian energy surface in $\mathbb{R}^4$ the Hamiltonian flow can never be minimal.

The next result by Dan looks at versions of the LeCalvez-Yoccoz Property in a symplectic setting.

Theorem (Cristofaro-Gardiner, Prasad (2024))

Let Y be a closed, connected, oriented 3-manifold equipped with a co-oriented contact structure ξ with torsion first Chern class. Let λ be any contact form defining ξ and let φ be the Reeb flow. Then for any proper compact invariant set $\Lambda \subset Y$, the complement $Y \setminus \Lambda$ is not minimal.

Theorem (Cristofaro-Gardiner, Prasad (2024))

Let Σ be a closed, oriented surface and let $\varphi : \Sigma \to \Sigma$ be any monotone area-preserving diffeomorphism. Then for any proper compact invariant set $\Lambda \subset \Sigma$ the complement $\Sigma \setminus \Lambda$ is not minimal.

The proofs of these theorems take up some innovations in the Fish-Hofer and extend the applicability considerably by novel constructions. Embedded contact homology an its structural properties provides "gradient lines", i.e. pseudoholomorphic curves with small transversal energy, but so-called Hofer energy blowing up and then the feral curve ideas suitably extended allow to squeeze out invariant subsets.

Packing Stability

Consider $M_k(b)$ the disjoint union of k Euclidean balls in $\mathbb{R}^4$ of radius $b > 0$. Introduce the ball-packing number

$$p_k(B_4(1)) := \sup_b \left\{ \frac{\mathrm{Vol}(M_k(b))}{\mathrm{Vol}(B_4(1))} \mid M_k(b) \text{ embedds symplectically } B_4(1) \right\}?$$

$$\begin{array}{ccccccccccccc} k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \dots \\ p_k & 1 & \frac{1}{2} & \frac{3}{4} & 1 & \frac{20}{25} & \frac{24}{25} & \frac{63}{64} & \frac{288}{289} & 1 & 1 & \end{array}$$

Theorem (Buse-Hind-Opshtein (2016))

Every closed symplectic four-manifold has packing stability.

The following is and answer to a question by Cristofaro-Gardiner and Hind.

Theorem (Edtmair (2025))

Any compact symplectic four-manifold with smooth boundary satisfies packing stability.

Packing Stability

Cristofaro-Gardiner and Hind recently identified the first examples of open symplectic manifolds of finite volume for which packing stability fails.

Theorem (Cristofaro-Gardiner, Hind (2023))

There exists an open bounded $U \subset \mathbb{R}^4$ diffeomorphic to the open 4-ball, such that $p_k(U) < 1$ for all $k \geq 1$.

The failure of packing stability in these examples is connected to the domains' "wild" boundary behavior. There is much more in the paper, for example something like a fractional symplectic Weyl theorem.
The phenomenon of packing stability is crucial for the sharpness of Weyl-type formulas, which describe the asymptotic growth of spectral invariants such as ECH capacities. Thus, symplectic ball packings form a bridge between geometric, topological, and spectral properties of symplectic manifolds.